Numerical evidence in favor of the Arenstorf formula

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Abstract

The formula $\lim_{N\to\infty}\sum_{p<N, p, p+2 \ both \ prime}\log(p)\log(p+2)=C_2$ is tested on the computer up to $N=2^{40}\approx 1.1\times 10^{12}$ and very good agreement is found.

The recent paper "There Are Infinitely Many Prime Twins" by R.F. Arenstorf [1] has raised a lot of excitement. The author claims to proved that:

$$\lim_{N\to\infty}\sum_{p<N, p, p+2 \ twins}\log(p)\log(p+2)=C_2 \tag{1}$$

where the twin constant

$$C_2\equiv 2\prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right)=1,32032363169373914785562\ldots. \tag{2}$$

Waiting for the formal approval of this result by mathematical community I have run the computer program to check the validity of (1). Even Hardy and Littlewood in their famous paper [2] have presented tables with numerical verification of their conjectures based on the existing that time data up to 9 000 000. Because the program I have written several years ago performs all operations on bits it was natural to store the data representing the actual arithmetical mean value of $\log(p)\log(p+2)$ at values of N forming the geometrical progression with the ratio 2, i.e. at $N=2^{22},2^{23},\ldots,2^{39},2^{40}$. The results are presented in the Table 1.

TABLE I

N	$1/N\sum_{p<N}\log(p)\log(p+2)$	$1/N\sum_{p<N}\log(p)\log(p+2)/C_2$
$2^{22}=4194304$	1.330875543	1.00799191245
$2^{23}=8388608$	1.325154123	1.00365856579
$2^{24}=16777216$	1.323501313	1.00240674443
$2^{25}=33554432$	1.320938577	1.00046575325
$2^{26}=67108864$	1.319310330	0.99923253570
$2^{27}=134217728$	1.320265943	0.99995630684
$2^{28}=268435456$	1.319095515	0.99906983694
$2^{29}=536870912$	1.319679380	0.99951205023
$2^{30}=1073741824$	1.320096901	0.99982827612
$2^{31}=2147483648$	1.320047000	0.99979048210
$2^{32}=4294967296$	1.320350510	1.00002035747
$2^{33}=8589934592$	1.320423613	1.00007572498
$2^{34}=17179869184$	1.320490713	1.00012654539
$2^{35}=34359738368$	1.320290443	0.99997486310
$2^{36}=68719476736$	1.320309503	0.99998929874
$2^{37}=137438953472$	1.320365187	1.00003147324
$2^{38}=274877906944$	1.320351882	1.00002139677
$2^{39}=549755813888$	1.320340769	1.00001297956
$2^{40}=1099511627776$	1.320322532	0.99999916675

As it can be seen from above table there is no apparent dependence on N. Indeed, trying to find heuristically the dependence on N we can argue that the probability to find twin pair around x is $C_2/\log^2(x)$ and hence the mean expectation value of the product $\log(p)\log(p+2)$ for p and $p+2$ on both sides of x ($p=x-1, p+2=x+1$) does not depend on x and we have simply that

$$\sum_{p<N,p,p+2\ twins}\log(p)\log(p+2)=C_2N \tag{3}$$

It can be contrasted with the calculation of the Brun constant

$$\mathcal{B}_2=\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\ldots<\infty. \tag{4}$$

The probability to find a pair of twins in the vicinity of x is $2C_2/\log^2(x)$, so the expected value of the finite approximation to the Brun constant can be estimated as follows:

$$\mathcal{B}_2(x)=\mathcal{B}_2(\infty)-\sum_{p\ \text{twin prime}>x}\frac{1}{p}\approx\mathcal{B}_2-4c_2\int_x^\infty\frac{du}{u\log^2(u)}=\mathcal{B}_2-\frac{4c_2}{\log(x)}. \tag{5}$$

It means that the plot of finite approximations $\mathcal{B}_2(x)$ to the original Brun constant is a linear function of $1/\log(x)$ [3] and from the partial sum $\mathcal{B}(x)$ calculated on the computer

up to x the limiting value can be extrapolated by adding $4C_2/\log(x)$: $\mathcal{B} = \mathcal{B}(x) + 4C_2/\log(x)$. To gain some idea what value of the limit can be the extrapolated from numbers in Table I the Figure 1 presents actual values of the mean value of $\log(p)\log(p+2)$ plotted against $1/N$. Fitting the straight line to these points by least square method gives the intercept (what corresponds to $N = \infty$) 1.3200385787619. In fact we see in Table I shortage of twins in the interval $(2^{26}, 2^{31})$ and in the next intervals some surplus of twins. Thus skipping the first 10 points and fitting straight line in the interval $(2^{32}, 2^{40})$ (in fact only two points are needed to determine straight line!) I got for the limiting value of the intercept 1.3203501777.

References

[1] R.F. Arenstorf, " There Are Infinitely Many Prime Twins", arXiv.math.NT/0405509

[2] G.H.Hardy and J.E. Littlewood, "Some problems of 'Partitio Numerorum' III: On the expression of a number as a sum of primes", *Acta Mathematatica* **44** (1922), p.1-70

[3] D.Shanks and J.W. Wrench, "Brun's Constant", Math. Comp. **28** (1974), p.293-299; M.Wolf, "Generalized Brun's constants", preprin IFTUWr 910/97, available at http://www.uni.wroc.pl/∼mwolf

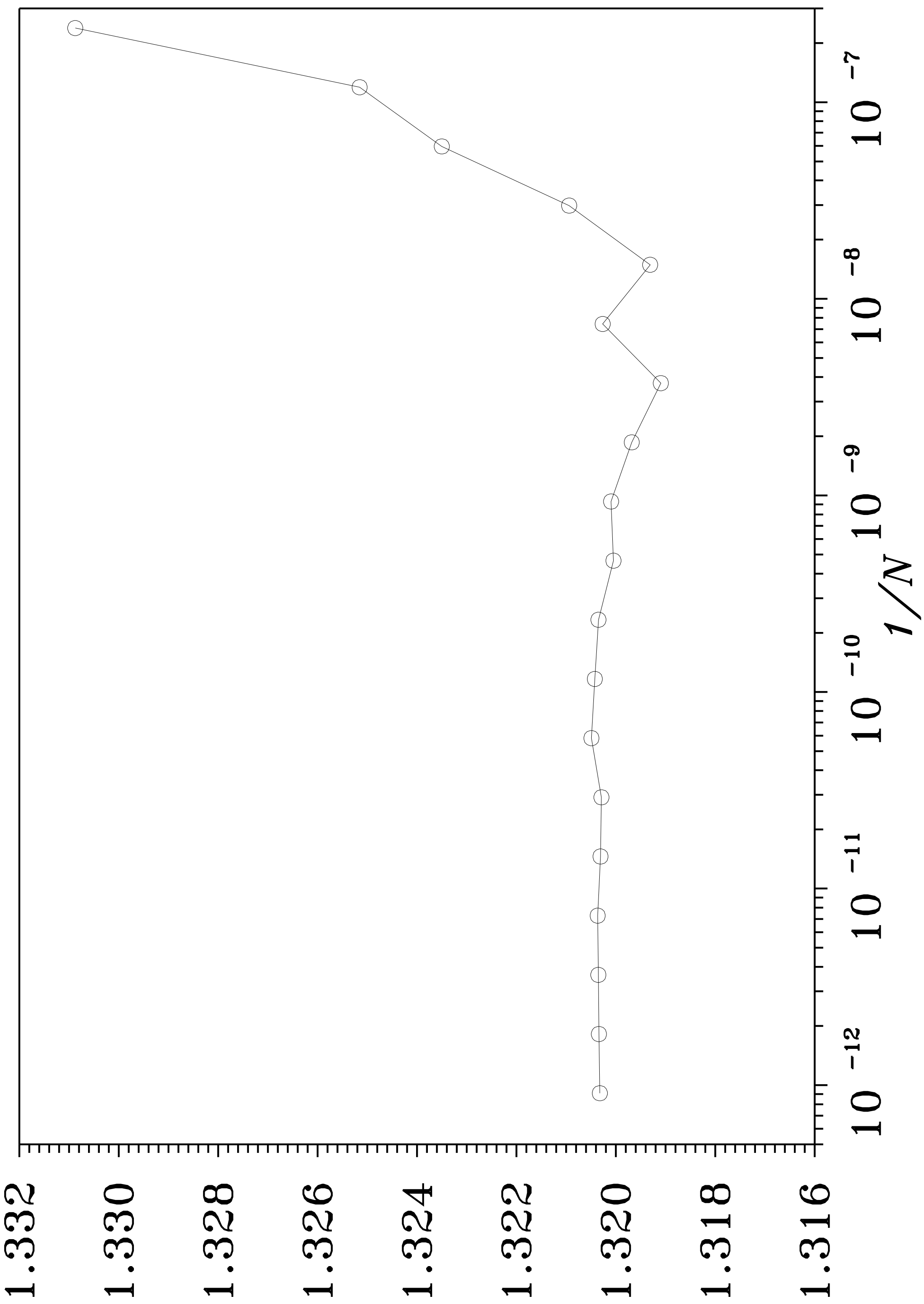

Figure 1: The plot of the actual mean values of $\log(p)\log(p+2)$ against $1/N$. Notice that on the x axis there is a logarithmic scale.